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Environmental influence on the measurement process in stochastic reduction models

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Abstract

We consider the energy-driven stochastic state vector reduction equation for the density matrix, which for pure state density matrices can be written in two equivalent forms. We use these forms to discuss the decoupling of the noise terms for independent subsystems, and to construct 'environmental' stochastic density matrices whose time-independent expectations are the usual quantum statistical distributions. We then consider a measurement apparatus weakly coupled to an external environment, and show that in mean field (Hartree) approximation the stochastic equation separates into independent equations for the apparatus and environment, with the Hamiltonian for the apparatus augmented by the environmental expectation of the interaction Hamiltonian. We use the Hartree approximated equation to study a simple accretion model for the interaction of the apparatus with its environment, as part of a more general discussion of when the stochastic dynamics predicts state vector reduction, and when it predicts the maintenance of coherence. We also discuss the magnitude of decoherence effects acting during the reduction process. Our analysis supports the suggestion that a measurement takes place when the different outcomes are characterized by sufficiently distinct environmental interactions for the reduction process to be rapidly driven to completion.

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1. Introduction

Understanding the measurement process has been a persistent problem since the inception of quantum mechanics. In the orthodox Copenhagen interpretation, measurements are accounted for by invoking a layer of classical, non-quantum mechanical reality; attempts to extend quantum mechanics to include the measuring apparatus itself lead to quandaries, such as the famous Schrödinger cat paradox. One approach to this problem that has been much studied recently [1–8] postulates that the Schrödinger equation is only an approximate description

of reality, and must be modified by small, nonlinear stochastic terms. These terms drive the state vector reduction process, and account for the non-observation of macroscopic quantum superpositions in measurement situations.

The proposal that a stochastic, nonlinear Schrödinger equation provides the phenomenology of quantum measurement passes a number of consistency tests. In its energy-driven form, it leads exactly to probabilities given by the Born rule [4, 8, 9], and for measurements on degenerate systems leads to the Lüders projection rule [9]. There are plausible arguments [7, 8], to be elaborated on here, that with a Planckian magnitude of the stochastic term, coherence is maintained where observed experimentally, while state vector reduction proceeds for measurement situations where discrete outcomes are observed. However, the stochastic Schrödinger equation is inherently non-relativistic [10, 11], involving the same stochastic differential at all spatial locations. This raises the issue of whether it is consistent with clustering—put simply, does the reduction of the state vector in a localized measuring apparatus proceed independently of what goes on far away from the laboratory? An affirmative answer to this question was given [8] under the assumption that all of the universe is governed by the pure state stochastic reduction equation. In this paper we extend this analysis in a number of directions, with the aim of understanding in greater detail the stochastic evolution of a 'measurement' system coupled to its environment.

Our discussion is organized as follows. In section 2 we give two equivalent forms of the Itô noise term in the stochastic evolution equation for a pure state density matrix, and use these to discuss clustering for disjoint subsystems. In section 3 we use one of these forms to prove the existence of pure state density matrices whose stochastic expectation gives the standard quantum statistical distributions. We also give a mixed state generalization of these results that is relevant when the Hamiltonian has degeneracies. In section 4 we consider a 'measurement' subsystem weakly coupled to an 'environment' subsystem, obeying overall the density matrix stochastic evolution equation, and derive the corresponding single system Hartree or mean field stochastic equations for the measurement and environment subsystems, working to first order accuracy in the interaction Hamiltonian.

In section 5 we give a survey of under what circumstances the stochastic evolution equation predicts state vector reduction, based on Planckian estimates for the magnitude of the stochastic term. We show that when the energy spread between superimposed states is small, coherence is maintained, in agreement with recent experiments on quantum coherence in large systems. On the other hand, when the energy fluctuations in a measurement system are large enough, state vector reduction proceeds rapidly to completion. We consider three types of energy fluctuations: thermal fluctuations, shot effect fluctuations in electric currents, and surface accretion fluctuations. For the latter, we use the mean field approximation of section 4 to construct a simple model for accretion processes, which motivates a quantitative discussion of their influence on the measurement process in both terrestrial and extraterrestrial environments. In section 6 we discuss the coexistence of standard decoherence mechanisms with the stochastic reduction process. In section 7 we state our conclusions regarding the implications of this analysis for the measurement process in quantum mechanics. In the appendix, we discuss a coherent state variant of the accretion model of section 5, in which reduction can proceed to coherent states.

2. Stochastic density matrix equations and clustering

We begin by recalling some formulae from the theory of stochastic Schrödinger equations [1–8]. Letting $|\chi\rangle$ be a normalized state vector, the standard stochastic evolution ('quantum state

diffusion') equation for $|\chi\rangle$ takes the form

$$\mathrm{d}|\chi\rangle = [\alpha\,\mathrm{d}t + \beta\,\mathrm{d}W_t]|\chi\rangle \tag{1a}$$

with $\mathrm{d}W_t$ a real Itô stochastic differential obeying

$$\mathrm{d}W_t^2 = \mathrm{d}t \qquad \mathrm{d}W_t\,\mathrm{d}t = 0 \tag{1b}$$

and with

$$\begin{aligned} \alpha &= -\mathrm{i}H - \tfrac{1}{8}\sigma^2[A - \langle A\rangle]^2 \\ \beta &= \tfrac{1}{2}\sigma[A - \langle A\rangle] \end{aligned} \tag{1c}$$

where σ is a numerical parameter governing the strength of the stochastic and drift terms, and A is a self-adjoint operator with expectation $\langle A\rangle$ in the state $|\chi\rangle$:

$$\langle A\rangle = \langle\chi|A|\chi\rangle. \tag{1d}$$

The energy-driven case of the stochastic evolution is obtained by taking A to be the Hamiltonian H, which we shall assume henceforth. It is straightforward to show that the evolution of equations (1a)–(1d) preserves the normalization of the state vector $|\chi\rangle$.

Defining the pure state density matrix $\rho = |\chi\rangle\langle\chi|$, it is easy to show that the state vector evolution of equations (1a)–(1d) implies that the density matrix evolution is given by

$$\mathrm{d}\rho = -\mathrm{i}[H, \rho]\,\mathrm{d}t - \tfrac{1}{8}\sigma^2[H, [H, \rho]]\,\mathrm{d}t + \tfrac{1}{2}\sigma N(\rho, H)\,\mathrm{d}W_t. \tag{2}$$

Direct calculation from equations (1a)–(1d) gives the coefficient $N(\rho, H)$ of the Itô noise term $\mathrm{d}W_t$ in equation (2) in the form

$$N(\rho, H) = \{\rho, H\} - 2\rho\,\mathrm{Tr}\,\rho H \tag{3a}$$

which by use of the pure state condition $\rho^2 = \rho$ can be written in the equivalent form

$$N(\rho, H) = [\rho, [\rho, H]]. \tag{3b}$$

Both of these forms have the property that if $\rho^2 = \rho$ then $\{\rho,\ \mathrm{d}\rho\} + (\mathrm{d}\rho)^2 = \mathrm{d}\rho$, which can be rewritten as $(\rho + \mathrm{d}\rho)^2 = \rho + \mathrm{d}\rho$, and so they preserve the pure state condition.

Let us now consider a system for which the Hamiltonian H is the sum of two Hamiltonians H_1, H_2 which depend on disjoint sets of variables, and investigate the conditions under which equations (2) and (3a), (3b) admit factorized solutions $\rho = \rho_1\rho_2$, with $\rho_{1,2}$ obeying equations of similar form driven by the respective Hamiltonians $H_{1,2}$. Substituting $H = H_1 + H_2$ and $\rho = \rho_1\rho_2$ into equations (3a), (3b), and using the facts that all variables in set 1 commute with all variables in set 2, and that $\mathrm{Tr} = \mathrm{Tr}_1\,\mathrm{Tr}_2$, we find, respectively, from equations (3a) and (3b) that

$$\begin{aligned} N(\rho_1\rho_2, H_1 + H_2) &= \rho_2[\{\rho_1, H_1\} - 2\rho_1\,\mathrm{Tr}_2\,\rho_2\,\mathrm{Tr}_1\,\rho_1 H_1] \\ &\quad + \rho_1[\{\rho_2, H_2\} - 2\rho_2\,\mathrm{Tr}_1\,\rho_1\,\mathrm{Tr}_2\,\rho_2 H_2] \end{aligned} \tag{4a}$$

$$N(\rho_1\rho_2, H_1 + H_2) = \rho_2^2[\rho_1, [\rho_1, H_1]] + \rho_1^2[\rho_2, [\rho_2, H_2]]. \tag{4b}$$

Clustering requires that

$$N(\rho_1\rho_2, H_1 + H_2) = \rho_2 N_1(\rho_1, H_1) + \rho_1 N_2(\rho_2, H_2) \tag{5}$$

with $N_{1,2}$ the restrictions of N to the 1, 2 subspaces. We see that equation (4a) obeys the clustering property by virtue of the trace conditions $\mathrm{Tr}_1\,\rho_1 = 1, \mathrm{Tr}_2\,\rho_2 = 1$, while equation (4$b$) satisfies the clustering property by virtue of the pure state conditions $\rho_1^2 = \rho_1$, $\rho_2^2 = \rho_2$.

Let us now examine the clustering properties of the remaining terms in equation (2). For the left-hand side, we find by use of the Itô extension of the chain rule that

$$\mathrm{d}(\rho_1\rho_2) = \rho_2\,\mathrm{d}\rho_1 + \rho_1\,\mathrm{d}\rho_2 + \mathrm{d}\rho_1\,\mathrm{d}\rho_2. \tag{6a}$$

Thus, in order to have $d\rho_1$ and $d\rho_2$ obeying equations of the same form as $d\rho$ but restricted to the 1, 2 subspaces, the left-hand side should take the form, using equations (1b) and (2),

$$d(\rho_1\rho_2) = \rho_2\, d\rho_1 + \rho_1\, d\rho_2 + \tfrac{1}{4}\sigma^2 N_1(\rho_1, H_1)N_2(\rho_2, H_2)\, dt. \tag{6b}$$

For the dt terms on the right-hand side of equation (2), we have

$$\begin{aligned} -i[H_1 + H_2, \rho_1\rho_2]\, dt - \tfrac{1}{8}\sigma^2[H_1 + H_2, [H_1 + H_2, \rho_1\rho_2]]\, dt \\ = \rho_2\{-i[H_1, \rho_1]\, dt - \tfrac{1}{8}\sigma^2[H_1, [H_1, \rho_1]]\, dt\} \\ + \rho_1\{-i[H_2, \rho_2]\, dt - \tfrac{1}{8}\sigma^2[H_2, [H_2, \rho_2]]\, dt\} - \tfrac{1}{4}\sigma^2[H_1, \rho_1][H_2, \rho_2]\, dt. \end{aligned} \tag{6c}$$

Assuming the conditions for the clustering property of equation (5) to hold for the Itô noise term, comparing equations (6a)–(6c) we see that the complete density matrix evolution equation will cluster if and only if

$$N_1(\rho_1, H_1)N_2(\rho_2, H_2) = -[H_1, \rho_1][H_2, \rho_2]. \tag{7}$$

This condition does not hold as in identity for either of the two possible forms for $N(\rho, H)$ given in equations (3a), (3b), and so the $\sigma^2\, dt$ or drift term in the stochastic evolution equation does couple disjoint systems.

However, there are two important special cases in which disjoint systems decouple asymptotically. The first of these cases corresponds [8] to taking $N(\rho, H)$ as in equation (3b), so that equation (7) becomes

$$[\rho_1, [\rho_1, H_1]][\rho_2, [\rho_2, H_2]] = -[H_1, \rho_1][H_2, \rho_2]. \tag{8a}$$

This equation is satisfied, by virtue of both the left- and right-hand sides vanishing, whenever either $[\rho_1, H_1] = 0$ or $[\rho_2, H_2] = 0$, conditions that are, respectively, obeyed when system 1 or 2 is at the endpoint of the state vector reduction process. In particular, if system 1 represents a measurement process, and system 2 represents a pure state environment at the endpoint of its reduction process, then the stochastic dynamics of system 1 are completely independent of the dynamics of its environment.

A more general case in which disjoint systems decouple asymptotically corresponds to taking $N(\rho, H)$ as in equation (3a), but not assuming the pure state condition so that this cannot be transformed to equation (3b). Equation (7) now becomes

$$[\{\rho_1, H_1\} - 2\rho_1 \operatorname{Tr}_1 \rho_1 H_1][\{\rho_2, H_2\} - 2\rho_2 \operatorname{Tr}_2 \rho_2 H_2] = -[H_1, \rho_1][H_2, \rho_2]. \tag{8b}$$

This equation is satisfied, again by virtue of both the left- and right-hand sides vanishing, whenever either ρ_1 is a linear combination of projectors on a degenerate submanifold of H_1, or ρ_2 is a linear combination of projectors on a degenerate submanifold of H_2. For example, in the latter case we would have $\rho_2 H_2 = H_2\rho_2 = E_2\rho_2$ for some degenerate submanifold energy E_2, together with $\operatorname{Tr}_2 \rho_2 = 1$, which imply the simultaneous vanishing of $\{\rho_2, H_2\} - 2\rho_2 \operatorname{Tr}_2 \rho_2 H_2$ and of $[H_2, \rho_2]$. Thus, if one were to adopt equations (2) and (3a) as a generalization of the density matrix evolution equation to the case of non-pure state density matrices, a pure state measurement process decouples from a mixed state environment whenever the density matrix for this environment is a linear combination of projectors on a degenerate submanifold of its Hamiltonian.

3. Martingale construction of the standard quantum statistical distributions

In order for the measurement system to decouple from its environment, we have seen that the environment must be described either by a pure state density matrix that commutes with the environment Hamiltonian, or by a mixed state density matrix that is a linear combination of

projectors on a degenerate submanifold of the environment Hamiltonian (with the second case equivalent to the first for a one-dimensional submanifold). This raises the question of how such a description can be compatible with the usual description of equilibrium environments in terms of the standard quantum statistical distributions, which are mixed state density matrices ρ obeying the trace condition $\mathrm{Tr}\,\rho = 1$, but which do not obey either the pure state condition $\rho^2 = \rho$ or the more general condition that the density matrix be a linear combination of projectors on a degenerate Hamiltonian submanifold. The answer is that in the theory of stochastic state vector reduction, the role of the usual mixed state density matrix is played [7] by the stochastic expectation $E[\rho]$ and not by ρ itself. Thus an equilibrium environment can be described by a stochastic density matrix that is a linear combination of projectors on a degenerate Hamiltonian submanifold, the stochastic expectation of which has the form $E[\rho] = f(H)$, with f one of the standard quantum statistical distribution functions of the Hamiltonian. Since equation (2) implies that $E[\rho]$ obeys the time evolution equation

$$\mathrm{d}E[\rho] = -\mathrm{i}[H, E[\rho]]\,\mathrm{d}t - \tfrac{1}{8}\sigma^2[H, [H, E[\rho]]]\,\mathrm{d}t \tag{9}$$

any $E[\rho]$ of the form $f(H)$ is time independent, as expected of the quantum statistical distributions.

To show that there are pure state density matrices with the required expectation, we proceed constructively by use of the density matrix evolution equation in the form

$$\mathrm{d}\rho = -\mathrm{i}[H, \rho]\,\mathrm{d}t - \tfrac{1}{8}\sigma^2[H, [H, \rho]]\,\mathrm{d}t + \tfrac{1}{2}\sigma[\{\rho, H\} - 2\rho\,\mathrm{Tr}\,\rho H]\,\mathrm{d}W_t. \tag{10}$$

Although we derived this equation in section 2 for pure state density matrices, we shall now use it, as suggested in the discussion associated with equation (8*b*), as a stochastic evolution equation for density matrices ρ that do not obey the pure state condition. Taking the initial ρ at time $t = 0$ as $\rho_0 = f(H)$, we see from equation (9), which follows by taking the expectation of equation (10), that $E[\rho] = f(H)$ for all times. Also, since equation (10) only involves the Hamiltonian H, the stochastically evolved ρ is still a function of H, and so commutes with H at all times. Thus, for the choice of initial condition $\rho_0 = f(H)$, equation (10) simplifies to

$$\mathrm{d}\rho = \tfrac{1}{2}\sigma[\{\rho, H\} - 2\rho\,\mathrm{Tr}\,\rho H]\,\mathrm{d}W_t. \tag{11}$$

Equation (11) defines ρ to be a martingale, for which the expectation E_s conditional on information available up to time s obeys $E_s[\rho_t] = \rho_s, s \leqslant t$, which reduces to $E[\rho] \equiv E_0[\rho_t] = \rho_0 = f(H)$ when $s = 0$. (Note that if instead of equation (10) we had used the stochastic equation obtained from equations (2) and (3*b*), the initial $\rho_0 = f(H)$ would not evolve in time at all, since equation (3*b*) vanishes identically when ρ commutes with H. This underscores again the fact that equations (3*a*) and (3*b*) are equivalent only for pure state density matrices, but define different stochastic evolutions for density matrices not obeying the pure state condition $\rho^2 = \rho$.)

Let us now show that at late times ρ evolves by equation (11) into a pure state projector when the Hamiltonian H is non-degenerate, or into a linear combination of projectors on a degenerate submanifold of H when H is degenerate. The proof of this parallels the proof [8,9] that equations (1*a*)–(1*d*) lead to state vector reduction. We consider the variance V of the Hamiltonian, defined by

$$V = \mathrm{Tr}\,\rho H^2 - (\mathrm{Tr}\,\rho H)^2 \tag{12a}$$

which by the Itô extension of the chain rule evolves in time as

$$\mathrm{d}V = \mathrm{Tr}\ \mathrm{d}\rho H^2 - 2\,\mathrm{Tr}\,\rho H\ \mathrm{Tr}\ \mathrm{d}\rho H - (\mathrm{Tr}\ \mathrm{d}\rho H)^2. \tag{12b}$$

Using equation (11) for $\mathrm{d}\rho$ to evaluate $\mathrm{Tr}\ \mathrm{d}\rho H^n$, we find

$$\mathrm{Tr}\ \mathrm{d}\rho H^n = \sigma[\mathrm{Tr}\,\rho H^{n+1} - \mathrm{Tr}\,\rho H^n\,\mathrm{Tr}\,\rho H]\,\mathrm{d}W_t. \tag{12c}$$

Thus, substituting equation (12c) for $n = 1,\ 2$ into equation (12b) and taking the expectation, we get

$$\mathrm{d}E[V] = -\sigma^2 E[V^2]\,\mathrm{d}t. \tag{13}$$

From here on the argument is identical to that of [8, 9], and leads to the conclusion that as $t \to \infty$ the variance V approaches 0 almost certainly. When the energy spectrum is non-degenerate, this implies that at late times only one density matrix element ρ_E is non-zero, and so the initial density matrix $\rho_0 = f(H)$ has evolved to a pure state density matrix obeying $\rho^2 = \rho$. More generally, when the energy spectrum is degenerate, the vanishing of the variance implies that the density matrix has evolved to a linear combination of projectors on a degenerate submanifold of the Hamiltonian. Thus, evolution of the initial density matrix $\rho_0 = f(H)$ by equation (10) leads to a late time density matrix that obeys $E[\rho] = f(H)$, and which is a pure state density matrix in the non-degenerate case, or a linear combination of projectors on a degenerate submanifold of H in the degenerate case. We take such density matrices as our model for the environment and, by the arguments of section 2, are assured that the evolution of measurement systems uncoupled by Hamiltonian interaction terms to this environment are independent of the environmental dynamics, when the total system evolves under the density matrix dynamics of equations (2) and (3a).

4. Mean field approximation for a system weakly coupled to its environment

Let us now consider two subsystems with disjoint variables that are weakly coupled through an interaction term ΔH in the Hamiltonian, so that the total Hamiltonian appearing in equation (10) is $H = H_1 + H_2 + \Delta H$. We shall take subsystem 1 to be a measuring apparatus (including the microscopic system being measured), whose reduction dynamics we wish to follow, while we take subsystem 2 to be the external environment with which this measuring apparatus interacts. We shall derive a mean field approximation to the dynamics, in which each subsystem obeys an independent system stochastic equation with a modified Hamiltonian, that reflects the mean interaction with the other subsystem. To this end, we substitute the independent subsystem Ansatz $\rho = \rho_1\rho_2$ into equation (10), and take the partial trace Tr_2 to average over the subsystem 2 dynamics, giving an effective equation for subsystem 1, and similarly, with the roles of 1 and 2 interchanged, to get an effective equation for subsystem 2. We shall assume that in the limit of vanishing coupling ΔH, the environment subsystem 2 is in one of the ensembles constructed in section 3 that is a function solely of H_2, so that in the presence of ΔH we have $[\rho_2, H_2] = \mathrm{O}(\Delta H)$. We do not make a corresponding assumption for subsystem 1, since we will be interested in the case in which this is initially in a generic pure state.

We proceed with this calculation term by term. From the left-hand side of equation (10), substituting equation (6a) we get

$$\mathrm{Tr}_2\ \mathrm{d}\rho = \mathrm{d}\rho_1\,\mathrm{Tr}_2\,\rho_2 + (\rho_1 + \mathrm{d}\rho_1)\,\mathrm{Tr}_2\,\mathrm{d}\rho_2 = \mathrm{d}\rho_1 \tag{14}$$

where we have used the condition $\mathrm{Tr}_2\,\rho_2 = 1$ which implies that $\mathrm{Tr}_2\ \mathrm{d}\rho_2 = 0$. From the first term on the right-hand side of equation (10) we get

$$\mathrm{Tr}_2(-\mathrm{i})[H, \rho]\,\mathrm{d}t = -\mathrm{i}[H_1, \rho_1]\,\mathrm{Tr}_2\,\rho_2 - \mathrm{i}\,\mathrm{Tr}_2[\Delta H, \rho_1\rho_2] - \mathrm{i}\rho_1\,\mathrm{Tr}_2[H_2, \rho_2]. \tag{15a}$$

The first term on the right of equation (15a) simply gives

$$-\mathrm{i}[H_1, \rho_1]\,\mathrm{d}t. \tag{15b}$$

Since $\mathrm{Tr}_2\,\Delta H\rho_2 = \mathrm{Tr}_2\,\rho_2\Delta H$, the second term on the right of equation (15a) becomes

$$-\mathrm{i}[\mathrm{Tr}_2\,\rho_2\Delta H, \rho_1]\,\mathrm{d}t \tag{15c}$$

and the third term on the right of equation (15a) vanishes. So in sum, the first term on the right-hand side of equation (10) gives

$$-\mathrm{i}[H_1 + (\mathrm{Tr}_2\,\rho_2\Delta H), \rho_1]\,\mathrm{d}t. \tag{16}$$

We turn next to the second term on the right-hand side of equation (10), which gives $-\frac{1}{8}\sigma^2\,\mathrm{d}t$ times the partial trace of the double commutator,

$$\begin{aligned}\mathrm{Tr}_2[H,[H,\rho]] &= \mathrm{Tr}_2[H_1+H_2+\Delta H,[H_1+H_2+\Delta H,\rho_1\rho_2]]\\ &= \mathrm{Tr}_2[H_1+\Delta H,[H_1+H_2+\Delta H,\rho_1\rho_2]]\\ &= \mathrm{Tr}_2\{[H_1+\Delta H,[H_1,\rho_1]\rho_2]+[H_1,[H_2,\rho_2]\rho_1]+[H_1,[\Delta H,\rho_1\rho_2]]\\ &\quad +[\Delta H,[H_2,\rho_2]\rho_1]+[\Delta H,[\Delta H,\rho_1\rho_2]]\}\\ &= [H_1+\mathrm{Tr}_2\,\rho_2\Delta H,[H_1+\mathrm{Tr}_2\,\rho_2\Delta H,\rho_1]]+\mathrm{O}((\Delta H)^2)\end{aligned} \tag{17}$$

where we have used the facts that (i) $\mathrm{Tr}_2[H_2, g_{1,2}] = 0$ for any operator $g_{1,2}$ acting on both subsystems 1 and 2, and that (ii) by our equilibrium assumption for the environment, $[H_2, \rho_2]$ is of order ΔH. (Step (ii) is the only one which does not go through in the corresponding effective equation calculation for the environment subsystem 2, leading to an additional term in its effective equation of motion given in equation (20a).)

Finally, we turn to the third term on the right-hand side of equation (10), which gives $\frac{1}{2}\sigma\,\mathrm{d}W_t$ times

$$\begin{aligned}\mathrm{Tr}_2[\{\rho,H\}-2\rho\,\mathrm{Tr}\,\rho H] &= \mathrm{Tr}_2[\{\rho_1\rho_2,H_1+H_2+\Delta H\}-2\rho_1\rho_2\,\mathrm{Tr}_1\,\mathrm{Tr}_2\,\rho_1\rho_2(H_1+H_2+\Delta H)]\\ &= \mathrm{Tr}_2[\{\rho_1\rho_2,H_1+\Delta H\}+\rho_1\{\rho_2,H_2\}-2\rho_1\rho_2\,\mathrm{Tr}_1\,\rho_1(H_1+\mathrm{Tr}_2\,\rho_2\Delta H)\\ &\quad -2\rho_1\rho_2\,\mathrm{Tr}_2\,\rho_2H_2] = \{\rho_1,H_1+\mathrm{Tr}_2\,\rho_2\Delta H\}-2\rho_1\,\mathrm{Tr}_1\,\rho_1(H_1+\mathrm{Tr}_2\,\rho_2\Delta H)\end{aligned} \tag{18}$$

where no approximations have been made.

Putting everything together, we see that the mean field approximation for the 'measurement' subsystem 1 is

$$\begin{aligned}\mathrm{d}\rho_1 = &-\mathrm{i}[H_1',\rho_1]\,\mathrm{d}t-\tfrac{1}{8}\sigma^2[H_1',[H_1',\rho_1]]\,\mathrm{d}t\\ &+\tfrac{1}{2}\sigma[\{\rho_1,H_1'\}-2\rho_1\,\mathrm{Tr}_1\,\rho_1H_1']\,\mathrm{d}W_t+\mathrm{O}(\sigma^2(\Delta H)^2\,\mathrm{d}t)\end{aligned} \tag{19a}$$

with the effective Hamiltonian:

$$H_1' = H_1 + \mathrm{Tr}_2\,\rho_2\Delta H. \tag{19b}$$

The corresponding equation for the 'environment' subsystem 2 is obtained by interchanging the labels 1 and 2, and restoring the term dropped in step (ii) leading to equation (17), giving

$$\begin{aligned}\mathrm{d}\rho_2 = &-\mathrm{i}[H_2',\rho_2]\,\mathrm{d}t-\tfrac{1}{8}\sigma^2[H_2',[H_2',\rho_2]]\,\mathrm{d}t+\tfrac{1}{2}\sigma[\{\rho_2,H_2'\}-2\rho_2\,\mathrm{Tr}_2\,\rho_2H_2']\,\mathrm{d}W_t\\ &-\tfrac{1}{8}\sigma^2[\mathrm{Tr}_1(\Delta H[H_1',\rho_1]),\rho_2]\,\mathrm{d}t+\mathrm{O}(\sigma^2(\Delta H)^2\,\mathrm{d}t)\end{aligned} \tag{20a}$$

with the effective Hamiltonian

$$H_2' = H_2 + \mathrm{Tr}_1\,\rho_1\Delta H. \tag{20b}$$

The added term on the second line of equation (20a) vanishes through order $(\Delta H)^2$ when the reduction process for subsystem 1 has concluded, since then the density matrix for subsystem 1 obeys $[H_1', \rho_1] = 0$ up to error terms of order $(\Delta H)^2$. As a consistency check on the calculation, we see that the mean field evolution equations obey $\mathrm{Tr}_1\,\mathrm{d}\rho_1 = \mathrm{Tr}_2\,\mathrm{d}\rho_2 = 0$, and so preserve the trace conditions $\mathrm{Tr}_1\,\rho_1 = \mathrm{Tr}_2\,\rho_2 = 1$.

5. Dynamics of the measurement process: when is coherence maintained, when does the state vector reduce?

Let us now examine the implications of equations (1)–(3) for measurements. We first have to specify the value of the parameter σ governing the magnitude of the stochastic process. If quantum mechanics is modified at all, it seems likely that such modifications come from new physics at the Planck scale, and so we adopt for this discussion the estimate [7,12] $\sigma \sim M_{\text{Planck}}^{-\frac{1}{2}}$, with M_{Planck} the Planck mass (in units with $\hbar = c = 1$). With this estimate, the reduction time t_R in seconds for a state with initial energy variance ΔE is given [7–9] by

$$t_R \sim \left(\frac{2.8\ \text{MeV}}{\Delta E}\right)^2 . \tag{21}$$

Thus, for ΔE equal to a proton mass, $t_R \sim 10^{-5}$ s, while for ΔE equal to the mass of a nitrogen molecule, one has $t_R \sim 10^{-8}$ s.

5.1. Maintenance of coherence

In order for stochastic energy-driven state vector reduction to give a viable phenomenology, it must satisfy the twin constraints of predicting the maintenance of coherence when this is observed, while predicting a rapid enough state vector reduction when a probabilistic choice between alternative outcomes is observed. We first discuss the constraints imposed by the maintenance of coherence. We begin by noting that according to equation (21), the sole criterion governing how rapidly the state vector reduces is the energy variance; whether the system is microscopic or macroscopic plays no role. Coherent superpositions of macroscopic states, involving large numbers of particles, will persist in time if the energy spread between the superimposed states is small enough. As a first example, consider the recent superconducting quantum interference device (SQUID) experiments [13,14] observing the existence of coherent superpositions of macroscopic states consisting of oppositely circulating supercurrents. Taking for discussion the experiment [13] (which of the two has the larger energy variance between the superimposed states), the energy spread ΔE is roughly 8.6×10^{-6} eV, and the circulating currents each correspond to the collective motion of $\sim 10^9$ Cooper pairs. According to equation (21), despite the macroscopic structure of the state vector, the state vector reduction time t_R for this experiment should be about 10^{23} s $\sim 3 \times 10^{15}$ years, and so maintenance of coherence is expected.

As our next example of the maintenance of coherence in macroscopic systems, we consider a recent experiment [15] demonstrating diffraction of the fullerenes C_{60} and C_{70}. We begin by noting that a diffraction pattern can be observed in a monoenergetic beam (in fact, this is the ideal condition for the experiment), so this class of experiments provides no evidence for coherent superpositions of states of differing energies. However, in a realistic experiment there will be an energy spread in the wavepacket for each particle constituting the beam. To see a diffraction pattern, the spread in de Broglie wavelengths $\Delta\lambda$ should be considerably smaller than λ; adopting the very weak bound $\Delta\lambda \leqslant \lambda$, we get the requirement that the spread in beam momenta Δp in each wavepacket should obey $\Delta p \leqslant p$. This implies that each wavepacket must have an energy spread ΔE obeying $\Delta E \leqslant 2E_{\text{kinetic}}$. In the experiments of [15] the beam was obtained from an oven at approximately 900 K, and so the the bound on the energy spread becomes $\Delta E \leqslant 2 \times (3/2) \times 900\ \text{K} \sim 0.23$ eV. The corresponding state vector reduction time predicted by equation (21) is of order 1.5×10^{14} s $\sim 5 \times 10^6$ years, and so energy-driven state vector reduction plays no role in this experiment. Similar estimates, and the same conclusion, would hold if larger objects, such as viruses, were diffracted.

These estimates suggest that in order to try to see the breakdown of coherence predicted by equation (21), one should consider experiments with systems having long-lived metastable states separated by a large energy gap from the ground state. In atomic systems, the requirements on stability of the metastable state are very severe, since for a typical atomic energy splitting of a few eV, equation (21) predicts a state vector reduction time of order 10^{12} s $\sim 3 \times 10^4$ years. For example, in the quantum intermittency experiments discussed in [16, 17], the metastable state lifetime is of order 1 s, and so stochastic state vector reduction effects are negligible. A potentially more promising case is provided by certain long-lived nuclear isomers [18], which are rendered metastable by their high spins, and which have large energy gaps from their ground states. For example, ^{178}Hf has an isomer with a half-life of 31 years suspended 2.4 MeV above its ground state. Quantum mechanics predicts that a coherent superposition of the isomeric state and the ground state should be stable for time intervals that are short relative to 31 years, whereas equation (21) predicts a spontaneous reduction of such a superposition to either the isomeric state or the ground state, with a reduction time of order 1 s. The only nuclear isomer to exist naturally on Earth is the metastable isomer of ^{180}Ta, which has a half-life of more than 10^{15} years, an energy gap of 75 keV from the ground state, and which accounts for roughly 1 part in 10^4 of naturally occurring tantalum. According to equation (21), a coherent superposition of the ground state and metastable isomer of ^{180}Ta should spontaneously reduce to either the isomeric state or the ground state, with a reduction time of order 23 min. Maintenance of coherence of such a superposition for times significantly longer than this would decisively rule out equations (1)–(3) as a phenomenology for state vector reduction. For example, if a laser using isomeric ^{180}Ta could be constructed, and if the characteristic relaxation times for conventional sources of dissipation could be made much longer than 23 min, then the effects of equation (21) might appear as an additional, unconventional source of stochastic fluctuations or of dissipation. It would clearly be of interest to work out the detailed implications of equations (1)–(3) for laser action in such a system.

5.2. Reduction in measurement situations

We turn now to the second requirement that must be satisfied by a phenomenology of state vector reduction, which is that it should lead to rapid reduction in experimental situations where a probabilistic outcome is observed. According to the von Neumann model for measurement [19], a measurement sets up a correlation between states $|f_\ell\rangle$ of a quantum system being measured, and macroscopically distinguishable states $|\mathcal{M}_\ell\rangle$ of the measuring apparatus $\mathcal{M}$, in such a way that an initial state

$$|f\rangle|\mathcal{M}_{\text{initial}}\rangle = \sum_\ell c_\ell |f_\ell\rangle|\mathcal{M}_{\text{initial}}\rangle \tag{22a}$$

evolves unitarily to

$$\sum_\ell c_\ell |f_\ell\rangle|\mathcal{M}_\ell\rangle. \tag{22b}$$

An objective state vector reduction model must then account for the selection of *one* of the alternatives $|f_\ell\rangle|\mathcal{M}_\ell\rangle$ from this superposition, with a probability given by $|c_\ell|^2$. If the energy spread among the states $|f_\ell\rangle$ is a typical atomic magnitude of a few eV, then as we have seen earlier using equation (21), the energy-driven model of equations (1)–(3) cannot quantitatively account for state vector reduction, unless the energy spreads among the alternative apparatus states in the superposition are much larger. Since in the ideal measurement model there is no energy transfer from the microscopic system to the apparatus, such an energy spread in the

measurement apparatus states can only be present if induced by environmental interactions, which are ignored in the von Neumann analysis. If these environmentally induced energy fluctuations are large enough for the state vector to reduce in a time much smaller than the measurement time, then the observed results will agree with the Copenhagen interpretation of the measurement process. If the reduction time were of the order of or larger than the measurement time, then equations (1)–(3) would predict stochastic fluctuations among alternative measurement outcomes, lasting until one is finally selected in a time roughly equal to t_R. However, as long as the apparatus states $|\mathcal{M}_\ell\rangle$ are orthogonal for different ℓ, no quantum interferences between different outcomes are possible.

To reiterate, for environmental interactions to be effective in producing state vector reduction, they must lead to energy fluctuations ΔE of the apparatus in the course of a measurement, that are large enough for equation (21) to predict a reduction time t_R that is less than the time it takes to make the measurement. Although different measuring devices have different response times, we shall assume for purposes of our discussion that relevant measurement times range down to 10^{-8} s, which requires for reduction a ΔE ranging up to $\sim$30 GeV. We shall consider three possible sources of energy fluctuations: thermal energy fluctuations, fluctuations in apparatus mass from particle accretion processes, and fluctuations in apparatus mass from amplified fluctuations in the currents that actuate the indicator devices.

Thermal energy and temperature fluctuations in a canonical ensemble, that is, with fixed particle number, are governed by the equations

$$\langle(\Delta E)^2\rangle_{AV} = k_{\mathrm{B}}T^2C_V \qquad \langle(\Delta T)^2\rangle_{AV} = k_{\mathrm{B}}T^2/C_V \tag{23}$$

with k_{B} Boltzmann's constant and with C_V the heat capacity. From these formulae, and the values of the heat capacity and thermal conductivity for various substances, together with the formulae governing the surface radiation rate, one can estimate that when a body is large enough for the energy fluctuations at room temperature to be of order 1–30 GeV, the thermal relaxation time over which such energy fluctuations occur is much larger than measurement times of interest. (For example, for 1 g of water at room temperature, the root mean square energy fluctuation is $\sim$14 GeV, and the thermal conduction relaxation time is $>$200 s.) The reason for this is that the rate for heat transfer processes is proportional to the temperature difference ΔT, and equation (23) shows that when a body is large, the temperature fluctuation ΔT is small. Thus we find that thermal energy fluctuations do not in general obey the criterion stated above for relevance to state vector reduction, that the energy fluctuation should occur within the measurement time.

A more significant source of energy fluctuations comes [7] from particle accretion processes, for which we formulate a simple model within the mean field framework of section 4. Consider a measuring apparatus which has N surface accretion sites for molecules of mass m. In Fock space representation, its Hamiltonian can be written as

$$H_1 = H_0 + \sum_{j=1}^{N} m a_j^\dagger a_j \tag{24a}$$

with H_0 the bulk Hamiltonian for the apparatus, and with $a_j^\dagger$ and a_j, respectively, the creation and annihilation operators for the accreted molecules. We assume the environment to contain a large number M of molecules that can be accreted onto the surface, with creation and annihilation operators $b_k^\dagger, b_k, k = 1, \ldots, M$, and with a coupling to the accretion sites given by

$$\Delta H = \sum_{j=1}^{N}\sum_{k=1}^{M}[A_{jk}a_j^\dagger b_k + \text{adjoint}]. \tag{24b}$$

This interaction Hamiltonian conserves the total number operator

$$N = \sum_{j=1}^{N} a_j^\dagger a_j + \sum_{k=1}^{M} b_k^\dagger b_k \tag{25}$$

in other words, the total number of molecules accreted onto the surface or remaining in the environment is constant.

In typical measurement situations, the environment density matrix will be diagonal in the number operator $\sum_{k=1}^{M} b_k^\dagger b_k$ of the molecules being accreted. In this case, which we term 'incoherent', the environmental expectation of ΔH vanishes

$$\mathrm{Tr}_2\, \rho_2 \Delta H = 0 \tag{26}$$

and the reduction process is governed, according to equations (19*a*) and (19*b*), by the measurement system Hamiltonian H_1 alone. (For a discussion of the coherent case, in which the environmental expectation of ΔH is non-zero, see the appendix.) The Hamiltonian ΔH still plays a role, since in order ΔH in probability amplitudes (corresponding to order $(\Delta H)^2$ in probabilities or transition rates) it leads to a sticking probability and an evaporation probability per unit time, respectively, for a molecule in the environment to accrete to the surface of the apparatus, and for a molecule already accreted to evaporate. As a result of these non-vanishing transition probabilities, the number of molecules accreted to the surface is constantly fluctuating. Assuming a simple colloid statistics model [20] in which each accretion site can hold only one molecule, the number of accreted molecules n obeys a Poisson distribution $\sigma(n, X) = \mathrm{e}^{-X} X^n / n!$ with the mean X proportional to the sticking probability and inversely proportional to the evaporation rate, and with the the root mean square fluctuation in the number of accreted molecules equal to the square root of the mean number X of accreted molecules.

Since distinguishable measurement outcomes must involve different configurations of the apparatus with respect to its environment, they will have different values of the accretion numbers $a_j^\dagger a_j$ associated with the N accretion sites. Thus, the energy eigenvalue H_1 of the measurement apparatus will differ for each distinguishable measurement outcome, with the spread of eigenvalues between any two outcomes being typically the mass of the accreted molecules m times the root mean square fluctuation in the number of accreted molecules. This statement assumes that the flux of accreting molecules in the environment is high enough for such a fluctuation to actually occur during the state vector reduction time. In estimating when this condition holds, we will follow the review of Redhead [21] in assuming that the sticking probability is of order unity, in which case the minimum time for one molecule to be accreted onto an area of 1 cm^2 can be read off from the molecular flux versus pressure tabulated in table 2 of [21]. At room temperature and atmospheric pressure (760 Torr) the time for one molecule to be accreted onto an area of 1 cm^2 is 3×10^{-24} s, while at an ultrahigh vacuum of 10^{-13} Torr it is 3×10^{-8} s. Thus, for an apparatus in the atmosphere at standard temperature and pressure, where the bulk of the accreting atoms are nitrogen molecules, the minimum apparatus area required for one molecule to accrete in a reduction time of 10^{-8} s (corresponding to a ΔE equal to the mass of a nitrogen molecule) is 3×10^{-16} cm^2, with the corresponding minimum area needed at a pressure of 10^{-13} Torr equal to 3 cm^2.

According to [21], the nighttime pressure at the surface of the Moon is about 10^{-13} Torr, while the pressure in interstellar space (within the galaxy) has been estimated as 10^{-18} Torr. Under the assumption of a sticking probability of order unity, the mass accretion rate scales as the pressure divided by the mean molecular velocity. While molecular velocities away from the vicinity of the Earth vary over a wide range, with effective temperatures in interstellar space ranging [22] from typically 50–100 to 10^6 K, we can get an estimate that is high by at most a factor of 2 or 3 by neglecting the velocity factor, and simply assuming that the mass

accretion rate scales with the pressure from the values given in the table of [21]. With this assumption, the minimum apparatus area needed for a reduction time of 10^{-8} s is 3 cm^2 at the surface of the Moon, and is 3×10^5 cm^2 = 30 m^2 in interstellar space. In intergalactic space, the predominant matter [22] is highly ionized hydrogen, with an effective temperature of order 10^4 K and a density of $\sim$0.23 proton m^{-3}. In this environment, the minimum apparatus area needed for a reduction time of 10^{-8} s is around 8×10^5 m^2 (corresponding to the accretion of 28 protons). If we were only to demand reduction in 3×10^{-4} s, then the needed apparatus area in interstellar space would decrease to less than 10 cm^2, while that in intergalactic space would decrease to $\sim$1 m^2. Thus, a capsule large enough to sustain Schrödinger's cat, situated in intergalactic space, would have a reduction time stimulated by collisions with molecules in the intergalactic medium much smaller than the length of time needed to ascertain whether the cat were dead or alive! Perhaps more to the point, in a typical high precision molecular beam experiment [23], the beam velocity is of order 10^5 cm s^{-1}, and the beam length is of order 2.7 m. Hence the time for the beam to traverse the apparatus is 2.7×10^{-3} s, and so the reduction time in intergalactic space for a capsule large enough to enclose the apparatus would be smaller, by at least an order of magnitude, than the measurement time. Clearly, in this situation the limits predicted by equation (21) are being pushed, and there could be realizable experiments which, in intergalactic space, would be predicted to start to show evidence of the stochastic fluctuation between outcomes characteristic of the time evolution of the state vector in stochastic reduction models. But it seems unlikely that such an experiment could be devised within the confines of the solar system—the ambient matter fluxes are too high.

In making some of the above estimates, it is convenient to have an alternative form of equation (21) that takes into account the accretion rate limit on ΔE, and which is derived as follows. Let $\mathcal{M}$ be the mass accretion rate per unit area of the apparatus in units s^{-1}, so that the mass accretion on area A in time t_R s is $\Delta E = \mathcal{A}\mathcal{M}t_R$. Substituting this into equation (21) and solving for t_R gives

$$t_R = \left(\frac{2.8 \text{ MeV}}{\mathcal{A}\mathcal{M}}\right)^{\frac{2}{3}}. \tag{27}$$

This formula can be used whenever at least one molecule is accreted in the time t_R. Given $\mathcal{M}$, we can calculate the area $\mathcal{A}$ corresponding to a given reduction time, and *vice versa*. For example, from equation (27) we find that for an apparatus of area 1 cm^2 in the atmosphere at standard temperature and pressure, the reduction time is $t_R = 5 \times 10^{-19}$ s, corresponding to the accretion of $\sim 1.5 \times 10^5$ molecules in the time t_R.

Throughout this analysis, we have assumed that the Hamiltonian that is relevant for the stochastic Schrödinger equation is the total Hamiltonian

$$H = \int \mathrm{d}^3x\, T_{00}(x) \tag{28}$$

defined by gravitational couplings to the stress-energy tensor $T_{\mu\nu}(x)$, which includes rest mass terms. Although in non-relativistic quantum mechanics one often drops rest mass terms when they lead to irrelevant constant energy shifts, there is no reason in principle to do so. In fact, in the standard model of elementary particles, all fermion rest masses arise from the Yukawa couplings of the fermions to the Higgs particle, so that from this point of view rest masses are not an additive constant term in the Hamiltonian, but are a dynamical product of interactions. We have also assumed that the relevant surface area is that of the whole apparatus, rather than just that of components of potentially small area such as solid state detectors, emulsions, or particle collector cups. This assumption is motivated by our decoupling analysis of section 2, where we saw that only non-interacting systems can be assumed (under certain equilibrium conditions) to decouple. The components of an apparatus (power supplies, magnets, vacuum

pumps, detectors, indicator pointers, magnetic recording domains) are not in equilibrium and are in interaction with one another, and so using the area of the whole apparatus, rather than of just the smallest components, seems justified.

We turn finally to a third potential source of energy fluctuations, arising from the amplified fluctuations in the currents which actuate experimental indicating or recording devices. Of course, if power sources are included, there are no overall current fluctuations, but power supplies are typically large in area and so when included in the system the accretion analysis just given indicates rapid reduction times. In a typical electrically amplified measurement, a final total charge transfer Ne (with e the charge of an electron) actuates an indicator or recording device. Assuming that the fluctuation in the current is the amplified fluctuation in the initially detected signal, for amplification gain G we have $\Delta N \sim G \times (N/G)^{\frac{1}{2}} = (NG)^{\frac{1}{2}}$, an estimate which agrees within factors of order unity with the standard noise estimate for photomultipliers [24]. Let us take N to correspond to a charge transfer of 1 mA (a voltage change of 10 V at 10 kΩ impedance) over a 10^{-8} s pulse, so that $N \sim 6 \times 10^7$, and assume a gain $G \sim 10^4$, giving $\Delta N \sim 8 \times 10^5$. Multiplying by the electron mass of 0.5×10^{-3} GeV, we find that the corresponding energy fluctuation is $\Delta E \sim 4 \times 10^2$ GeV, which leads to state vector reduction in 5×10^{-11} s. Thus, electric current fluctuations play a significant role in state vector reduction when the 'apparatus' is defined to exclude power sources.

Our overall conclusion is that conditions under which laboratory experiments are performed, as well as conditions under which space capsule experiments might be performed in the foreseeable future, are consistent with state vector reduction times as estimated by equation (21) that are well within experimental measurement times.

5.3. Experiments with semi-silvered mirrors

In the preceding two subsections we have considered the case in which the energy variance is so small that coherence is maintained, and the case in which the energy variance is large enough that reduction proceeds rapidly to completion. Let us now briefly consider a case that contains elements of both, in which an apparatus is constructed using semi-silvered mirrors (for photons) or thin detectors (for particles), so that there is a probability amplitude α for no interaction with the apparatus and the maintenance of coherence, and a probability amplitude β for a measurement to take place. The state vector then has the form, after the measurement interaction,

$$\alpha|f\rangle|\mathcal{M}_{\text{initial}}\rangle + \beta \sum_{\ell} c_\ell |f_\ell\rangle|\mathcal{M}_\ell\rangle. \tag{29}$$

Assuming that $|\mathcal{M}_{\text{initial}}\rangle$ and $|\mathcal{M}_\ell\rangle$ differ sufficiently in energy for reduction to take place, there are now two possible classes of outcomes. With probability $|\beta c_\ell|^2$ the ℓth measurement outcome is observed, while with probability $|\alpha|^2$ the initial state is unchanged, corresponding to transmission through the semi-silvered mirror or thin detector. In making the latter assertion, we are using the fact that, as shown in [9], the model of equations (1)–(3) obeys the Lüders projection postulate. That is, a component of the wavefunction lying within a submanifold of Hilbert space that is energy degenerate (or nearly degenerate, in the sense of section 5.1) survives unchanged in form, with the appropriate probability, as an outcome of the reduction process. This corresponds exactly to what happens when a beam is transmitted through a partially silvered mirror or a thin detector. This discussion generalizes immediately to the case in which the transmitted beam has different phase shifts in the various terms in the superposition over wavefunction components $|f_\ell\rangle$.

6. Coexistence of reduction and decoherence

In the previous section, we considered the effects on an apparatus of inelastic collisions, in which the mass of the apparatus fluctuates. However, an apparatus considerably more frequently suffers elastic collisions with atoms and photons in its environment, which are responsible for the decoherence effects [25–27] that have been much discussed in the literature. We shall argue in this section that decoherence effects do not substantially modify the results of the preceding section.

We begin with a general argument that is independent of the details of modeling decoherence. When elastic interactions with the environment are taken into account, the effective apparatus wavefunction has to be extended to include the wavefunctions of all particles with which it interacts during the reduction time t_R. Since this extension of the apparatus definition increases its area, the rate of mass fluctuations is increased, and the effective reduction time estimated in the preceding section is, if anything, decreased. (Because t_R is in general shorter than the time for molecules or photons in the environment to collide with one another, we do not have to continue this enlargement of the apparatus another step to include the particles with which the decohering particles interact—they can be regarded as effectively non-interacting. For example, we estimated earlier that in air at standard temperature and pressure, the reduction time t_R for an apparatus of area 1 cm^2 is $\sim 5 \times 10^{-19}$ s (during which time it accretes $\sim 1.5 \times 10^5$ air molecules), whereas the mean time between collisions of air molecules with each other is $\sim 10^{-10}$ s. Since the reduction time through accretion scales as the inverse $\frac{2}{3}$ power of the density of the environmental medium, while the time between collisions of a molecule scales inversely as the density, in more dilute environments this inequality gets stronger.) With the definition of the apparatus wavefunction extended in this way, equation (22*b*) is modified to read

$$\sum_\ell c_\ell |f_\ell\rangle |\mathcal{M}_\ell\rangle |\Psi_{e\ell}\rangle \tag{30}$$

with $|\Psi_{e\ell}\rangle$ the environmental wavefunction associated with the ℓth apparatus state. We can now apply the analysis developed earlier, using equations (1)–(3) to describe the stochastic reduction of the wavefunction of equation (30), with the conclusions reached previously unaltered. The wavefunction of the extended apparatus remains unit normalized, and its density matrix remains a pure state density matrix, which stochastically evolves to an energy eigenstate with the rate given by equation (21).

Decoherence effects manifest themselves by the exponential decay with time of the inner product $\langle \Psi_{e\ell} | \Psi_{e\ell'} \rangle$, for $\ell \neq \ell'$, so that the environmental states associated with different measurement outcomes become rapidly orthogonal. Correspondingly, in the reduced density matrix for the original unextended apparatus, obtained by tracing out the environmental degrees of freedom, there is an exponential decay of the off-diagonal matrix elements. For the superposition of energy eigenstates that is relevant for our discussion, the relevant decay or decoherence rate of the off-diagonal reduced matrix element is given by [25, 26]

$$D = N_{\text{scatt}} \text{Re}\,[1 - \langle S_\ell^\dagger S_{\ell'} \rangle]. \tag{31a}$$

Here N_{scatt} is the number of scatterings by environmental particles in unit time, $\langle \cdots \rangle$ is an expectation in the state of the scattering particle, and S_ℓ and $S_{\ell'}$ are the scattering matrices acting on this particle when it scatters on the respective components of the apparatus wavefunction with state labels ℓ and ℓ'. In our context, these apparatus states differ only by the addition of some number of accreted molecules, and so in a weak scattering approximation the product $S_\ell^\dagger S_{\ell'}$ can be approximated as S_M, with S_M the scattering matrix for an environmental particle

to scatter from the accreted molecules. In this approximation, equation (31) simplifies to

$$D = N_{\text{scatt}}\,\text{Re}\,[1 - \langle S_M \rangle]. \tag{31b}$$

Expressing the scattering matrix S_M in terms of the corresponding scattering amplitude, and using the optical theorem, equation (31*b*) reduces [25] to

$$D = \tfrac{1}{2} \times \text{scattering rate}. \tag{32}$$

Combining equation (32) with our estimates earlier for an area 1 cm^2 apparatus in air, the decoherence rate arising from molecules of the environment scattering on the molecules accreted during the reduction time is $\frac{1}{2} \times 10^{10}\ \text{s}^{-1} \times 1.5 \times 10^5 \sim 0.7 \times 10^{15}\ \text{s}^{-1}$, several orders of magnitude smaller than the reduction rate $t_R^{-1} \sim 2 \times 10^{18}\ \text{s}^{-1}$, and so decoherence effects are in fact unimportant over the duration of the reduction process. Since the ratio of the reduction rate to the decoherence rate (calculated for the number of molecules accreted over the reduction time) scales with area as $\mathcal{A}^{\frac{1}{3}}$, this conclusion remains true down to an apparatus area of $\sim 4 \times 10^{-11}$ cm^2, where the two rates become approximately equal. Thus, even for a very small apparatus the environmental states $|\Psi_{e\ell}\rangle$ are nearly identical to the initial environmental state $|\Psi_e\rangle$, and so the wavefunction of equation (30) is negligibly entangled with the environment. Therefore instead of using the extended apparatus to discuss the reduction process, one is justified in ignoring decoherence and using the original unextended apparatus as in equation (22*b*). Our conclusion here for energy-driven reduction models differs significantly from that reached [28,29] for spontaneous localization models, where decoherence effects over the reduction time are substantial; however, in these models the various apparatus states in the superposition of equation (22*b*) differ by a displacement of the center of mass of some part of the apparatus, which gives a decoherence effect proportional to the (macroscopic) scattering cross section of that part of the apparatus which is displaced.

7. Discussion and conclusions

The analysis we have given of a number of aspects of the effect of the environment on the measurement process, including the decoupling of isolated systems from environments in equilibrium, the effect of energy fluctuations induced by mass accretion, and the effect of decoherence processes, supports the view that the energy-driven stochastic Schrödinger equation gives a viable phenomenology of state vector reduction. According to this picture, a measurement takes place when the different outcomes are characterized by sufficiently large environmentally induced energy fluctuations in the apparatus for the state vector reduction process, which is driven by the energy variance, to proceed rapidly to completion. The infinite Von Neumann regression (of an apparatus measuring an apparatus measuring an apparatus ..., ad infinitum) terminates when the apparatus size is large enough for its energy fluctuations to lead to state vector reduction within the specified observation time. This requirement on apparatus size meshes in a natural way with the intuitively obvious requirement that, in a measurement, different experimental outcomes must be macroscopically distinguishable.

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Appendix. Coherent case of the accretion model

Anandan [30] has raised the interesting question of whether state vector reduction can proceed to a coherent state endpoint. One way in which this can happen is when the signal amplification process involves coherent states, as discussed in [27]. Another way, which we shall discuss here, corresponds to the 'coherent' case of the accretion model formulated in section 5.2, in which the environment is in a coherent state, so that the environmental expectation of ΔH is non-zero. Assuming for simplicity that there is only one accretion site, which can be multiply occupied, we have then

$$H_1 + \mathrm{Tr}_2\, \rho_2 \Delta H = H_0 + m a_1^\dagger a_1 + \lambda a_1^\dagger + \lambda^* a_1 \tag{A.1a}$$

with λ given by

$$\lambda = \sum_{k=1}^{M} A_{1k}\, \mathrm{Tr}_2\, \rho_2 b_k. \tag{A.1b}$$

Assuming H_0 to commute with a_1, equations (A.1a) and (A.1b) describe the zero forcing frequency limit of the forced harmonic oscillator, which has been extensively studied [31,33], and can be succinctly solved by coherent state methods [32,33]. Defining z and c_1 by

$$z \equiv -\lambda/m \qquad c_1 \equiv a_1 - z = a_1 + \lambda/m \tag{A.2a}$$

we have

$$H_1 = H_0 + m c_1^\dagger c_1 + \text{constant} \tag{A.2b}$$

which in its c_1 dependence is a standard harmonic oscillator. The c_1 oscillator ground state $|0\rangle$ obeys

$$c_1|0\rangle = 0 \Rightarrow a_1|0\rangle = z|0\rangle \tag{A.2c}$$

in other words, $|0\rangle$ is a coherent state in terms of the original operators a_1.

Ignoring an overall constant arising from terms in equation (A.2b) that commute with a_1, the general eigenstate of equation (A.2b) is $|n\rangle$, with n the number of c_1 quanta, and has energy eigenvalue mn. This state is a coherent superposition of states with different numbers of molecules on the accretion site. For energy eigenvalue n, the probability $P(n|k)$ of finding $n-k$ molecules on the site can be exactly expressed [31,33] as a Laguerre polynomial, and for $|z| \ll 1$ and n large can be approximated [34] as

$$P(n|k) \simeq [J_{|k|}(2n^{\frac{1}{2}}|z|)]^2 \tag{A.3a}$$

with J_k the order k Bessel function; the Bessel function addition formula [35]

$$1 = J_0(w)^2 + 2\sum_{n=1}^{\infty} J_n(w)^2 \tag{A.3b}$$

implies that the probabilities of equation (A.3a) sum to unity:

$$\sum_{k=-\infty}^{\infty} P(n|k) = 1. \tag{A.3c}$$

Equation (A.3a) is rapidly oscillating as a function of k, but using the asymptotic estimate [36]

$$J_\nu(\nu \sec\beta) \simeq \left(\frac{2}{\pi\nu\tan\beta}\right)^{\frac{1}{2}} \cos\left(\nu\tan\beta - \nu\beta - \frac{1}{4}\pi\right) \tag{A.4a}$$

it is easily seen that the averaged envelope of $P(n|k)$ is given by

$$\overline{P(n|k)} \simeq \frac{1}{\pi}\frac{1}{(4n|z|^2 - k^2)^{\frac{1}{2}}} \tag{A.4b}$$

showing that the values of k are mainly distributed (apart from an exponentially decaying tail) between $-2n^{\frac{1}{2}}|z|$ and $2n^{\frac{1}{2}}|z|$.

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